Mathematical Proof of Perpetual Calendar

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Suppose $y, m, and\ d$ are arbitrary integers, where $1 \leq y \leq 9999$, $1 \leq m \leq 12$, and $1 \leq d \leq 31$.

Let y', m' be

$$y' = \begin{cases} y & \text{when } 3 \leq m \\ y-1 & \text{when } 3 \nleq m \end{cases}$$

and

$$m' = \begin{cases} m & \text{when } 3 \leq m \\ m+12 & \text{when } 3 \nleq m \end{cases}$$

Assume q_i, r_i, $i = 1, 2, \cdots, 6$ are integers where

$$y' = 2000q_1 + r_1, \qquad 0 \leq r_1 \leq 2000,$$

$$r_1 = 400q_2 + r_2, \qquad 0 \leq r_2 \leq 400,$$

$$r_2 = 100q_3 + r_3, \qquad 0 \leq r_3 \leq 100,$$

$$r_3 = 20q_4 + r_4, \qquad 0 \leq r_4 \leq 20,$$

$$r_4 = 4q_5 + r_5, \qquad 0 \leq r_5 \leq 4,$$

$$r_5 = 1q_6 + r_6, \qquad 0 \leq r_6 \leq 1 \ .$$

Further, assume u_i, v_i, $i = 1, 2$ are integers where

$$d = 7u_1 + v_1, \qquad 0 \le v_1 \le 7,$$

$$v_1 = 1u_2 + v_2, \qquad 0 \le v_2 \le 1.$$

In addition, we define function $f: \{1, 2, \cdots, 12\} \to \{0,1,\cdots,6\}$ to be

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 0 & 3 & 2 & 5 & 0 & 3 & 5 & 1 & 4 & 6 & 2 & 4 \end{pmatrix}$$

Then, at this time, the following statements hold:

(i) $0 \le q_1 < 5, 0 \le q_2 < 5, 0 \le q_3 < 4, 0 \le q_4 < 5, 0 \le q_5 < 5, 0 \le q_6 < 4$

(ii) $0 \le u_1 < 5, 0 \le u_2 < 7$

(iii) $q_1 = 4$ for some y, m, d

(iv) $q_2 = 4$ for some y, m, d

(v) $q_3 = 3$ for some y, m, d

(vi) $q_4 = 4$ for some y, m, d

(vii) $q_5 = 4$ for some y, m, d

(viii) $q_6 = 3$ for some y, m, d

(ix) $u_1 = 4$ for some y, m, d

(x) $u_2 = 6$ for some y, m, d

(xi) $f(m) + 0u_1 + 1u_2 + 0q_1 + 0q_2 + 5q_3 + 4q_4 + 5q_5 + 1q_6 \equiv y' + \left\lfloor \frac{y'}{4} \right\rfloor - \left\lfloor \frac{y'}{100} \right\rfloor + \left\lfloor \frac{y'}{400} \right\rfloor + \left\lfloor \frac{13m'+8}{5} \right\rfloor + d \pmod 7$

(The right side of Equation (xi) is the Zeller formula).

Proof First, we prove Statement (i). When $3 \le m$, then by definition, $y' = y$. Then, because $0 \le 1 \le y \le 9999 < 10000$, Statement (i) is true by Lemma 2. When $3 \nleq m$, then by definition, $y' = y - 1$. Then, because $0 \le y - 1 \le 9998 < 10000$, Statement (i) is true by Lemma 2. Statement (ii) holds true by Lemma 2 because $0 \le 1 \le d \le 31 < 32$. Next, we prove Statement (iii). When $y = 8000$, $m = 3$, $d = 1$, then by definition, $y' = y = 8000$ and the following holds:

$$8000 = 2000 \cdot 4 + 0, \qquad 0 \le 0 < 2000,$$

$$0 = 400 \cdot 0 + 0, \qquad 0 \le 0 < 400,$$

$$0 = 100 \cdot 0 + 0, \qquad 0 \le 0 < 100,$$

$$0 = 20 \cdot 0 + 0, \qquad 0 \le 0 < 20,$$

$$0 = 4 \cdot 0 + 0, \qquad 0 \le 0 < 4,$$

$$0 = 1 \cdot 0 + 0, \qquad 0 \le 0 < 1.$$

Hence, q_1 = 4. In Statement (iv), when y = 1600, m = 3, d = 1, then by definition, y' = y = 1600, and the following holds:

$$1600 = 2000 \cdot 0 + 1600, \qquad 0 \le 1600 < 2000,$$

$$1600 = 400 \cdot 4 + 0, \qquad 0 \le 0 < 400,$$

$$0 = 100 \cdot 0 + 0, \qquad 0 \le 0 < 100,$$

$$0 = 20 \cdot 0 + 0, \qquad 0 \le 0 < 20,$$

$$0 = 4 \cdot 0 + 0, \qquad 0 \le 0 < 4,$$

$$0 = 1 \cdot 0 + 0, \qquad 0 \le 0 < 1.$$

Hence, q_2 = 4. Similarly, for Statement (v), we consider the case where y = 300, m = 3, d = 1; for Statement (iv), the case where y = 80, m = 3, d = 1; for Statement (vii), where y = 16, m = 3, d = 1; and for Statement (viii), where y = 3, m = 3, d = 1. In addition, for Statement (ix), we consider the case where y = 1, m = 1, d = 28; and for Statement (x), where y = 1, m = 1, d = 6. Finally, we prove Statement (xi). First, by Lemma 1, the following is true:

$$0q_1 + 0q_2 + 5q_3 + 4q_4 + 5q_5 + 1q_6 \equiv y' + \left\lfloor \frac{y'}{4} \right\rfloor - \left\lfloor \frac{y'}{100} \right\rfloor + \left\lfloor \frac{y'}{400} \right\rfloor \quad (mod\ 7)$$

In addition, we can easily verify the following:

$$f(m) \equiv \left\lfloor \frac{13m' + 8}{5} \right\rfloor \qquad (mod\ 7)$$

When we consider

$$0u_1 + 1u_2 \equiv 7u_1 + 1u_2 = d \quad (mod\ 7)\ ,$$

then the following statement holds true:

$$f(m) + 0u_1 + 1u_2 + 0q_1 + 0q_2 + 5q_3 + 4q_4 + 5q_5 + 1q_6$$

$$= f(m) + (0u_1 + 1u_2) + (0q_1 + 0q_2 + 5q_3 + 4q_4 + 5q_5 + 1q_6)$$

$$\equiv \left\lfloor \frac{13m' + 8}{5} \right\rfloor + d + \left(y' + \left\lfloor \frac{y'}{4} \right\rfloor - \left\lfloor \frac{y'}{100} \right\rfloor + \left\lfloor \frac{y'}{400} \right\rfloor \right)$$

$$= y' + \left\lfloor \frac{y'}{4} \right\rfloor - \left\lfloor \frac{y'}{100} \right\rfloor + \left\lfloor \frac{y'}{400} \right\rfloor + \left\lfloor \frac{13m' + 8}{5} \right\rfloor + d \quad (mod\ 7)$$

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Lemma **1**. Let x be an arbitrary integer where $0 \le x$. Suppose $q_i, r_i, i = 1,2, \dots ,6$ are integers where

$$x = 2000q_1 + r_1, \qquad 0 \le r_1 < 2000,$$

$$r_1 = 400q_2 + r_2, \qquad 0 \le r_2 < 400,$$

$$r_2 = 100q_3 + r_3, \qquad 0 \le r_3 < 100,$$

$$r_3 = 20q_4 + r_4, \qquad 0 \le r_4 < 20,$$

$$r_4 = 4q_5 + r_5, \qquad 0 \le r_5 < 4,$$

$$r_5 = 1q_6 + r_6, \qquad 0 \le r_6 < 1\ .$$

Then at this time, the following holds:

$$0q_1 + 0q_2 + 5q_3 + 4q_4 + 5q_5 + 1q_6 \equiv x + \left\lfloor \frac{x}{4} \right\rfloor - \left\lfloor \frac{x}{100} \right\rfloor + \left\lfloor \frac{x}{400} \right\rfloor \quad (mod\ 7)$$

Proof Suppose $o_i,\ p_i, i = 1, 2, 3, 4$ are integers where

$$x = 400o_1 + p_1, \qquad 0 \le p_1 < 400,$$

$$x = 100o_2 + p_2, \qquad 0 \le p_2 < 100,$$

$$x = 4o_3 + p_3, \qquad 0 \le p_3 < 4,$$

$$x = 1o_4 + p_4, \qquad 0 \le p_4 < 1\ .$$

Then, at this time, the following are true:

$$x = 400o_1 + p_1$$

and

$$x = 2000q_1 + 400q_2 + r_2\ .$$

Therefore, $p_1 \equiv x \equiv r_2 \,(mod\ 400)$. Then, because $0 \le p_1 < 400,\ 0 \le r_2 < 400,$ we can say $p_1 = r_2$. So,

$$400o_1 = 2000q_1 + 400q_2 .$$

Therefore,

$$o_1 = 5q_1 + q_2\ .$$

In addition,

$$x = 100o_2 + p_2$$

and

$$x = 2000q_1 + 400q_2 + 100q_3 + r_3 .$$

Hence, $p_2 \equiv x \equiv r_3 (mod\ 100)$. Then, because $0 \leq p_2 < 100,\ 0 \leq r_3 < 100$, we can say $p_2 = r_3$ holds true. Thus,

$$100o_2 = 2000q_1 + 400q_2 + 100q_3\ .$$

Therefore,

$$o_2 = 20q_1 + 4q_2 + q_3\ .$$

In addition,

$$x = 4o_3 + p_3$$

and

$$x = 2000q_1 + 400q_2 + 100q_3 + 20q_4 + 4q_5 + r_5\ .$$

Therefore, $p_3 \equiv x \equiv r_5\ \ (mod\ 4)$. Then, because $0 \leq p_3 < 4,\ 0 \leq r_5 < 4$, we can say $p_3 = r_5$ holds true. Thus,

$$4o_3 = 2000q_1 + 400q_2 + 100q_3 + 20q_4 + 4q_5$$

and

$$o_3 = 500q_1 + 100q_2 + 25q_3 + 5q_4 + q_5\ .$$

Then,

$$x = o_4$$

and

$$x = 2000q_1 + 400q_2 + 100q_3 + 20q_4 + 4q_5 + 1q_6\ .$$

Hence,

$$o_4 = 2000q_1 + 400q_2 + 100q_3 + 20q_4 + 4q_5 + 1q_6 \ .$$

Therefore, the following statements hold true:

$$x + \left\lfloor \frac{x}{4} \right\rfloor - \left\lfloor \frac{x}{100} \right\rfloor + \left\lfloor \frac{x}{400} \right\rfloor$$

$$= o_4 + o_3 - o_2 + o_1$$

$$= (2000q_1 + 400q_2 + 100q_3 + 20q_4 + 4q_5 + 1q_6) + (500q_1 + 100q_2 + 25q_3 + 5q_4 + q_5) - (20q_1 + 4q_2 + q_3) + (5q_1 + q_2)$$

$$= (2000 + 500 - 20 + 5)q_1 + (400 + 100 - 4 + 1)q_2 + (100 + 25 - 1)q_3 + (20 + 5)q_4 + (4 + 1)q_5 + 1q_6$$

$$= 2485q_1 + 497q_2 + 124q_3 + 25q_4 + 5q_5 + 1q_6$$

$$\equiv 0q_1 + 0q_2 + 5q_3 + 4q_4 + 5q_5 + 1q_6 \quad (mod\ 7)$$

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Lemma **2**. Let x, y be arbitrary integers where $0 \leq x < 10000,\ 0 \leq y < 32.$
Suppose $q_i, r_i, i = 1, 2, \ldots, 6$ are integers where

$$x = 2000q_1 + r_1, \quad 0 \leq r_1 < 2000,$$

$$r_1 = 400q_2 + r_2, \quad 0 \leq r_2 < 400,$$

$$r_2 = 100q_3 + r_3, \quad 0 \leq r_3 < 100,$$

$$r_3 = 20q_4 + r_4, \quad 0 \leq r_4 < 20,$$

$$r_4 = 4q_5 + r_5, \quad 0 \le r_5 < 4,$$

$$r_5 = 1q_6 + r_6, \quad 0 \le r_6 < 1$$

Suppose $u_i, v_i, i = 1, 2$ are integers where

$$y = 7u_1 + v_1, \quad 0 \le v_1 < 7,$$

$$v_1 = 1u_2 + v_2, \quad 0 \le v_2 < 1$$

Then, at this time, the following statements hold:

(i) $0 \le q_1 < 5,\ 0 \le q_2 < 5,\ 0 \le q_3 < 4,\ 0 \le q_4 < 5,\ 0 \le q_5 < 5,\ 0 \le q_6 < 4.$

(ii) $0 \le u_1 < 5,\ 0 \le u_2 < 7.$

Proof First, we prove Statement (i). Supposing $q_1 < 0$, then $2000q_1 \le -2000$ and $2000q_1 + 2000 \le 0$ because $q_1 \le -1$. On the other hand, there is a contradiction because $0 \le x = 2000q_1 + r_1 < 2000q_1 + 2000$ holds true. Hence, $0 \le q_1$. Similarly, the following statement holds true: $0 \le q_i, i = 2,3, \dots ,6$. In addition, supposing $5 \le q_1$, then $10000 \le 2000q_1$. On the other hand, there is a contradiction because $2000q_1 \le 2000q_1 + r_1 = x < 10000$ holds true. Hence, $q_1 < 5$. Similarly, the following is true: $q_2 < 5, q_3 < 4, q_4 < 5, q_5 < 5, q_6 < 4$. We have proven Statement (i), and similarly we can prove Statement (ii). ■