Transformation of Zeller formula

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Let y, m, and d be arbitrary integers that satisfy $0 \le y$, $3 \le m \le 14$ and $1 \le d \le 31$. Let C, C′, C″, and N be arbitrary integers such that

$$y = 100\mathrm{C}' + \mathrm{N}, \qquad 0 \le \mathrm{N} < 100,$$
$$\mathrm{C}' = 100\mathrm{C}'' + \mathrm{C}, \qquad 0 \le \mathrm{C} < 100.$$

Then, we prove that

$$5(\mathrm{C}\%4) + 5\lfloor \mathrm{N}/4 \rfloor + (\mathrm{N}\%4) + \left\lfloor \frac{13m+8}{5} \right\rfloor + d$$
$$\equiv y + \left\lfloor \frac{y}{4} \right\rfloor - \left\lfloor \frac{y}{100} \right\rfloor + \left\lfloor \frac{y}{400} \right\rfloor + \left\lfloor \frac{13m+8}{5} \right\rfloor + d \quad (mod\ 7)$$

(right-hand side is the Zeller formula). Here, for integers a and b, $b \ne 0$, the remainder after dividing a by b is written as a%b.

[Proof]

Given integers q_i and r_i, $i = 1,2,3,4$ satisfying

$$y = 400q_1 + r_1, \qquad 0 \le r_1 < 400,$$
$$r_1 = 100q_2 + r_2, \qquad 0 \le r_2 < 100,$$
$$r_2 = 4q_3 + r_3, \qquad 0 \le r_3 < 4,$$
$$r_3 = 1q_4 + r_4, \qquad 0 \le r_4 < 1,$$

we show that

$$146097q_1 + 36524q_2 + 1461q_3 + 365q_4 = 365y + \left\lfloor \frac{y}{4} \right\rfloor - \left\lfloor \frac{y}{100} \right\rfloor + \left\lfloor \frac{y}{400} \right\rfloor. \qquad (1)$$

Given $s_i, t_i \in \mathbb{Z}, i = 1,2,3,4$ satisfying

$$y = 400s_1 + t_1, \qquad 0 \le t_1 < 400,$$
$$y = 100s_2 + t_2, \qquad 0 \le t_2 < 100,$$
$$y = 4s_3 + t_3, \qquad 0 \le t_3 < 4,$$
$$y = 1s_4 + t_4, \qquad 0 \le t_4 < 1,$$

both s_1 and q_1 are the quotient of y divided by 400, and therefore $s_1 = q_1$. Further,

$$y = 100s_2 + t_2$$

and

$$y = 400q_1 + 100q_2 + r_2.$$

Therefore, $t_2 \equiv y \equiv r_2$ (mod 100), and since $0 \le t_2 < 100$, $0 \le r_2 < 100$, we have $t_2 = r_2$. Thus,

$$100s_2 = 400q_1 + 100q_2$$

and therefore

$$s_2 = 4q_1 + q_2.$$

Next, we have

$$y = 4s_3 + t_3$$

and

$$y = 400q_1 + 100q_2 + 4q_3 + r_3.$$

Therefore, $t_3 \equiv y \equiv r_3$ (mod 4), and since $0 \le t_3 < 4$, $0 \le r_3 < 4$ we have $t_3 = r_3$.
Thus,

$$4s_3 = 400q_1 + 100q_2 + 4q_3$$

and

$$s_3 = 100q_1 + 25q_2 + q_3.$$

Then,

$$y = s_4$$

and

$$y = 400q_1 + 100q_2 + 4q_3 + 1q_4,$$

therefore

$$s_4 = 400q_1 + 100q_2 + 4q_3 + 1q_4.$$

Thus,

$$365y + \left\lfloor \frac{y}{4} \right\rfloor - \left\lfloor \frac{y}{100} \right\rfloor + \left\lfloor \frac{y}{400} \right\rfloor$$

$$= 365s_4 + s_3 - s_2 + s_1$$

$$= 365(400q_1 + 100q_2 + 4q_3 + 1q_4) + (100q_1 + 25q_2 + q_3) - (4q_1 + q_2) + q_1$$

$$= (365 \cdot 400 + 100 - 4 + 1)q_1 + (365 \cdot 100 + 25 - 1)q_2 + (365 \cdot 4 + 1)q_3 + (365 \cdot 1)q_4$$

$$= 146097q_1 + 36524q_2 + 1461q_3 + 365q_4$$

which is (1).

Next, we prove the following:

$$\text{C\%4} = q_2, \qquad \lfloor \text{N}/4 \rfloor = q_3, \qquad \text{N\%4} = q_4. \qquad (2)$$

First, $C' = 100C'' + C$ and therefore $C' \equiv C$ (*mod* 4). Moreover, $C' = s_2 = 4q_1 + q_2$, which gives $C' \equiv q_2\ (mod\ 4)$, and therefore we get $C \equiv C' \equiv q_2$ (*mod* 4). On the other hand, it is easily understood that $0 \le q_2 < 4$. Thus, C%4 $= q_2$. Further, $N = t_2 = r_2$, and therefore $\lfloor N/4 \rfloor = \lfloor r_2/4 \rfloor = q_3$. Moreover, $N\%4 = r_2\%4 = r_3 = q_4$.

From (1) and (2), we therefore get

$$y+\left\lfloor\frac{y}{4}\right\rfloor-\left\lfloor\frac{y}{100}\right\rfloor+\left\lfloor\frac{y}{400}\right\rfloor \equiv 365y+\left\lfloor\frac{y}{4}\right\rfloor-\left\lfloor\frac{y}{100}\right\rfloor+\left\lfloor\frac{y}{400}\right\rfloor$$

$$= 146097q_1 + 36524q_2 + 1461q_3 + 365q_4 \equiv 5q_2 + 5q_3 + q_4$$

$$= 5(\mathrm{C}\%4) + 5\lfloor \mathrm{N}/4 \rfloor + (\mathrm{N}\%4) \qquad (mod\ 7). \qquad (3)$$

From (3) we immediately get

$$5(\mathrm{C}\%4) + 5\lfloor \mathrm{N}/4 \rfloor + (\mathrm{N}\%4) + \left\lfloor\frac{13m+8}{5}\right\rfloor + d$$

$$\equiv y+\left\lfloor\frac{y}{4}\right\rfloor-\left\lfloor\frac{y}{100}\right\rfloor+\left\lfloor\frac{y}{400}\right\rfloor+\left\lfloor\frac{13m+8}{5}\right\rfloor + d \qquad (mod\ 7).$$

(QED)