Calculation by an Iterative Division Algorithm using the Fairfield and Zeller Formulas

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Kentaro SUZUKI

Consider y, m, and d as random integers such that $0 \leq y$, $3 \leq m \leq 14$, and $1 \leq d \leq 31$.

$$\begin{aligned} y &= 400q_1 + r_1, & 0 &\leq r_1 < 400, \\ r_1 &= 100q_2 + r_2, & 0 &\leq r_2 < 100, \\ r_2 &= 40q_3 + r_3, & 0 &\leq r_3 < 40, \\ r_3 &= 20q_4 + r_4, & 0 &\leq r_4 < 20, \\ r_4 &= 8q_5 + r_5, & 0 &\leq r_5 < 8, \\ r_5 &= 4q_6 + r_6, & 0 &\leq r_6 < 4, \\ r_6 &= 1q_7 + r_7, & 0 &\leq r_7 < 1, \end{aligned}$$

where $q_i, r_i \in Z, i = 1,2,\cdots,7$,

$$\begin{aligned} m - 3 &= 5s_1 + t_1, & 0 &\leq t_1 < 5, \\ t_1 &= 2s_2 + t_2, & 0 &\leq t_2 < 2, \\ t_2 &= 1s_3 + t_3, & 0 &\leq t_3 < 1, \end{aligned}$$

and where $s_i, t_i \in Z, i = 1,2,3$,

$$\begin{aligned} d - 1 &= 7u_1 + v_1, & 0 &\leq v_1 < 7, \\ v_1 &= 1u_2 + v_2, & 0 &\leq v_2 < 1, \end{aligned}$$

and where $u_i, v_i \in Z, i = 1,2$.

We have to prove that

$$0q_1 + 5q_2 + 1q_3 + 4q_4 + 3q_5 + 5q_6 + 1q_7 + 6s_1 + 5s_2 + 3s_3 + 0u_1 + 1u_2 + 3$$
$$\equiv y + \left\lfloor \frac{y}{4} \right\rfloor - \left\lfloor \frac{y}{100} \right\rfloor + \left\lfloor \frac{y}{400} \right\rfloor + \left\lfloor \frac{13m+8}{5} \right\rfloor + d \quad (mod\ 7)$$

(the right-hand side of this equation is the **Zeller formula**).

[**Proof**]

First, let us prove that

$$146097q_1 + 36524q_2 + 14610q_3 + 7305q_4 + 2922q_5 + 1461q_6 + 365q_7 + 153s_1 + 61s_2 + 31s_3 + 7u_1 + 1u_2 - 305$$
$$= 365y + \left\lfloor \frac{y}{4} \right\rfloor - \left\lfloor \frac{y}{100} \right\rfloor + \left\lfloor \frac{y}{400} \right\rfloor + \left\lfloor \frac{306(m+1)}{10} \right\rfloor + d - 428 \qquad (1)$$

(the right-hand side of this equation is the **Fairfield formula**). To do this, we need to prove that

$$146097q_1 + 36524q_2 + 14610q_3 + 7305q_4 + 2922q_5 + 1461q_6 + 365q_7$$
$$= 365y + \left\lfloor\frac{y}{4}\right\rfloor - \left\lfloor\frac{y}{100}\right\rfloor + \left\lfloor\frac{y}{400}\right\rfloor \qquad (1.1).$$

If $o_i, p_i \in Z, i = 1,2,3,4$

$$y = 400o_1 + p_1, \quad 0 \le p_1 < 400,$$
$$y = 100o_2 + p_2, \quad 0 \le p_2 < 100,$$
$$y = 4o_3 + p_3, \quad 0 \le p_3 < 4,$$
$$y = 1o_4 + p_4, \quad 0 \le p_4 < 1,$$

and both o_1 and q_1 are the quotients obtained by dividing y by 400; therefore , $o_1 = q_1$.

Further,

$$y = 100o_2 + p_2$$

and

$$y = 400q_1 + 100q_2 + r_2$$

; therefore, $y \equiv p_2 \equiv r_2 \ (mod\ 100)$. Moreover, since $0 \le p_2 < 100,\ 0 \le r_2 < 100,\ p_2 = r_2$.

Thus,

$$100o_2 = 400q_1 + 100q_2.$$

Consequently,

$$o_2 = 4q_1 + q_2.$$

Further,

$$y = 4o_3 + p_3$$

and

$$y = 400q_1 + 100q_2 + 40q_3 + 20q_4 + 8q_5 + 4q_6 + r_6$$

; therefore, $y \equiv p_3 \equiv r_6 \ (mod\ 4)$, and since $0 \le p_3 < 4,\ 0 \le r_6 < 4$, it follows that $p_3 = r_6$.

Thus,

$$4o_3 = 400q_1 + 100q_2 + 40q_3 + 20q_4 + 8q_5 + 4q_6$$

and

$$o_3 = 100q_1 + 25q_2 + 10q_3 + 5q_4 + 2q_5 + q_6.$$

Next,

$$y = o_4$$

and

$$y = 400q_1 + 100q_2 + 40q_3 + 20q_4 + 8q_5 + 4q_6 + 1q_7$$

; hence,

$$o_4 = 400q_1 + 100q_2 + 40q_3 + 20q_4 + 8q_5 + 4q_6 + 1q_7.$$

Consequently,

$$\begin{aligned} &365y + \left\lfloor\frac{y}{4}\right\rfloor - \left\lfloor\frac{y}{100}\right\rfloor + \left\lfloor\frac{y}{400}\right\rfloor \\ &= 365o_4 + o_3 - o_2 + o_1 \\ &= 365(400q_1 + 100q_2 + 40q_3 + 20q_4 + 8q_5 + 4q_6 + 1q_7) \\ &\quad + (100q_1 + 25q_2 + 10q_3 + 5q_4 + 2q_5 + q_6) - (4q_1 + q_2) + q_1 \\ &= (365\cdot 400 + 100 - 4 + 1)q_1 + (365\cdot 100 + 25 - 1)q_2 + (365\cdot 40 + 10)q_3 \\ &\quad + (365\cdot 20 + 5)q_4 + (365\cdot 8 + 2)q_5 + (365\cdot 4 + 1)q_6 + (365\cdot 1)q_7 \\ &= 146097q_1 + 36524q_2 + 14610q_3 + 7305q_4 + 2922q_5 + 1461q_6 + 365q_7 \end{aligned}$$

which proves (1.1). It then follows that

$$153s_1 + 61s_2 + 31s_3 = \left\lfloor\frac{306(m+1)}{10}\right\rfloor - 122 \qquad (1.2).$$

This can be easily confirmed as follows. Consider $m = 11$; then,

$$\begin{aligned} 153s_1 + 61s_2 + 31s_3 &= 153\cdot 1 + 61\cdot 1 + 31\cdot 1 = 153 + 61 + 31 = 245 = 367 - 122 \\ &= \left\lfloor\frac{3672}{10}\right\rfloor - 122 = \left\lfloor\frac{306\cdot 12}{10}\right\rfloor - 122 = \left\lfloor\frac{306(11+1)}{10}\right\rfloor - 122 \\ &= \left\lfloor\frac{306(m+1)}{10}\right\rfloor - 122. \end{aligned}$$

Since (1.1) and (1.2) hold,

$$\begin{aligned} &146097q_1 + 36524q_2 + 14610q_3 + 7305q_4 + 2922q_5 + 1461q_6 + 365q_7 + 153s_1 + 61s_2 \\ &\quad + 31s_3 + 7u_1 + 1u_2 - 305 \\ &= (146097q_1 + 36524q_2 + 14610q_3 + 7305q_4 + 2922q_5 + 1461q_6 + 365q_7) \\ &\quad + (153s_1 + 61s_2 + 31s_3) + (7u_1 + 1u_2) - 305 \\ &= \left(365y + \left\lfloor\frac{y}{4}\right\rfloor - \left\lfloor\frac{y}{100}\right\rfloor + \left\lfloor\frac{y}{400}\right\rfloor\right) + \left(\left\lfloor\frac{306(m+1)}{10}\right\rfloor - 122\right) + (d-1) - 305 \end{aligned}$$

$$= 365y + \left\lfloor \frac{y}{4} \right\rfloor - \left\lfloor \frac{y}{100} \right\rfloor + \left\lfloor \frac{y}{400} \right\rfloor + \left\lfloor \frac{306(m+1)}{10} \right\rfloor + d - 428,$$

which proves (1). It is well known that

$$365y + \left\lfloor \frac{y}{4} \right\rfloor - \left\lfloor \frac{y}{100} \right\rfloor + \left\lfloor \frac{y}{400} \right\rfloor + \left\lfloor \frac{306(m+1)}{10} \right\rfloor + d - 428$$

$$\equiv y + \left\lfloor \frac{y}{4} \right\rfloor - \left\lfloor \frac{y}{100} \right\rfloor + \left\lfloor \frac{y}{400} \right\rfloor + \left\lfloor \frac{13m+8}{5} \right\rfloor + d \quad (mod\ 7).$$

Therefore, from (1),

$$0q_1 + 5q_2 + 1q_3 + 4q_4 + 3q_5 + 5q_6 + 1q_7 + 6s_1 + 5s_2 + 3s_3 + 0u_1 + 1u_2 + 3$$

$$\equiv 146097q_1 + 36524q_2 + 14610q_3 + 7305q_4 + 2922q_5 + 1461q_6 + 365q_7 + 153s_1 + 61s_2 + 31s_3 + 7u_1 + 1u_2 - 305$$

$$= 365y + \left\lfloor \frac{y}{4} \right\rfloor - \left\lfloor \frac{y}{100} \right\rfloor + \left\lfloor \frac{y}{400} \right\rfloor + \left\lfloor \frac{306(m+1)}{10} \right\rfloor + d - 428$$

$$\equiv y + \left\lfloor \frac{y}{4} \right\rfloor - \left\lfloor \frac{y}{100} \right\rfloor + \left\lfloor \frac{y}{400} \right\rfloor + \left\lfloor \frac{13m+8}{5} \right\rfloor + d \quad (mod\ 7).$$

(QED)